Some positive and negative results on the strong comparison principle for degenerate elliptic and parabolic problems

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Abstract

The weak and strong comparison principles for elliptic and parabolic partial differential equations play an important role in the existence and uniqueness theory for nonlinear degenerate problems of these types. In a few rather general situations we will give an affirmative answer, whereas in some special cases simple counterexamples will be constructed. We make use of some classical techniques combined with the most recent results for divergence-type problems with a linear drift term. Practical applications include hydrology, oil exploration, and spread of nitrates.

Of course, the problems of validity of weak and strong comparison principles are quite technical and require very delicate hypotheses. We will show a simple example of a nonlinear degenerate problem with the p-Laplacian in a ball with a radially symmetric solution that obeys the ***weak*** comparison principle if the "spectral" parameter $\lambda > 0$ is sufficiently large (***the larger*** $\lambda > 0$ ***is, the better***). In contrast, if $\lambda > 0$ is large enough, say $\lambda > \lambda_p > 0$, then the ***strong*** comparison principle is false. This simple (counter-)example shows that the hypotheses that are "good" for the weak comparison principle might be "bad" for the strong comparison principle.

Nevertheless, in proving the ***strong*** comparison principle we will treat only those cases in which it is possible, under some additional hypotheses, to derive the strong comparison principle from the weak one. This is the usual strategy in practically all methods I know about.

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