Department of Mathematics & Statistics

Ph.D admission written test

Time: 90 Minutes **May 11, 2017**

Total Marks: 105

NAME:____

Instructions

1. Write your name in **CAPITAL** letters.
2. We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}[i]$ the set of natural numbers, integers, rational numbers, real numbers, complex numbers and Gaussian integers respectively.

   For $n \geq 1$, the set $\mathbb{Z}_n$ denotes the set $\mathbb{Z}/n\mathbb{Z}$ and S_n denotes the permutation group on n-symbols.

   We denote by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, the unit disc in $\mathbb{C}$.
3. There is a provision for partial marking for questions in section 3.
4. There are three sections. The first section is True or false and the second section is fill in the blanks.
   - In the first section, every correct answer will be awarded 3 marks and a wrong answer will be awarded -3 marks.
   - In the second section correct answer for every blank carries 3 marks. (i.e., If there are k blanks in a question, it will carry $3k$ marks.)
5. The third section has one or more correct answers. In this section
   - each question has four choices.
   - if a wrong answer is selected in a question then that entire question will carry 0 marks.
   - the candidate gets full credit of 6 marks, only if he/she selects all the correct answers and no wrong answers. 3 marks will be awarded for an answer to a question if it is partially correct and a wrong answer is not selected.
6. These question-cum-answer-sheets must be returned to the invigilator before leaving the examination hall.

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1 True/False [15 marks]

1. When interpolating a continuous function by a polynomial at equally spaced points on a given interval, the polynomial interpolant always converges pointwise to the function as the number of interpolation points increases.

2. If a non-singular symmetric matrix is not positive definite, then it can not have a Cholesky factorization.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) := x^2 - y^2$. Then the point $(0, 0)$ is a saddle point of the function f.

4. Let $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : \|x\| < 1\}$, then there exists at least one solution $u \in C^2(\bar{\Omega})$ to the problem

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u(x_1, x_2) = \frac{x_1^2 - x_2^2}{3} \quad \text{on} \quad \partial\Omega$$

   with $u(0, 0) = 1$.

5. Let $f : \mathbb{C} \to \mathbb{C}$ be the map given by $f(z) := \sin z - z$. Then the image of f is $\mathbb{C}$.

2 Fill in the blanks [24 marks]

1. Suppose that the fixed point iteration

$$x_{m+1} = \frac{x_m(x_m^2 + 15)}{3x_m^2 + 5}, \quad m = 0, 1, \ldots$$

   converges to some $\alpha > 0$ for a suitable x_0. Then α is __________ and the order of convergence is __________.

2. For an infinitely differentiable function f and $h > 0$, if the approximate derivative

$$D_h f(x) = \frac{\alpha f(x) + \beta(f(x+h) - f(x-h)) + \gamma(f(x+2h) - f(x-2h))}{h}$$

   yields error $f'(x) - D_h f(x) = Ch^4$, then α is __________ , β is __________ and γ is __________.

3. Let a and b be two positive real numbers and $(x_n)_{n=1}^{\infty}$ be the sequence defined by $x_n := (a^n + b^n)^{\frac{1}{n}}$ for n in $\mathbb{N}$. Then $\lim_{n\to\infty} x_n =$ __________ .

4. Let $C[0,1]$ denote the set of all continuous real valued functions on the interval $[0,1]$. Let $T : (C[0,1], \|\,.\,\|_\infty) \to \mathbb{R}$ be defined by

$$T(f) = \int_0^1 t f(t) dt$$

for all $f \in C[0,1]$. Then $||T|| =$ ____________.

5. Let $S := \{h : \mathbb{D} \to \mathbb{D} : h$ is analytic in $\mathbb{D}$ such that $h(z)^2 = \overline{h(z)}$ for all $z \in \mathbb{D}\}$. Then the cardinality of $S =$ ____________.

3 Questions with one or more correct answers [66 marks]

1. Let $n \geq 1$ and $\mathcal{P}_n := \{a_0 + a_1 X + \cdots + a_n X^n : a_i \in \mathbb{R}\}$ denote the set of all polynomials of degree at most n. If $Q(f) = (f(0) + 3f(1/3) + 3f(2/3) + f(1))/8$ is a quadrature rule for approximation of $I(f) = \int_0^1 f(x)dx$, then $I(f) - Q(f) = 0$ for all f in
   (a) $\mathcal{P}_1$ (b) $\mathcal{P}_2$ (c) $\mathcal{P}_3$ (d) $\mathcal{P}_4$

2. Given a convex function u on the open interval (a,b) which of the following statements are true:
   - (a) $\frac{u(d)-u(c)}{d-c} \geq \frac{u(e)-u(d)}{e-d}$ provided $a < c < d < e < b$.
   - (b) $\frac{u(d)-u(c)}{d-c} \leq \frac{u(e)-u(d)}{e-d}$ provided $a < c < d < e < b$.
   - (c) u is Lipschitz continuous in $[c,d] \subset (a,b)$ for $a < c \leq d < b$.
   - (d) u may be a nowhere differentiable function in $(c,d) \subset (a,b)$.

3. Let $u(x) = x^2$ and $v(x) = x|x|$ for x in $\mathbb{R}$. Which of the following are true?
   - (a) The functions u and v are linearly dependent.
   - (b) The functions u and v are linearly independent.
   - (c) The functions u and v are solutions of a second order linear homogeneous ODE.
   - (d) The Wronskian of u and v is zero at every point x in $\mathbb{R}$.

4. Which of the following maps are constant?
   - (a) $f : \mathbb{D} \to \mathbb{C}$ such that f is analytic and $f(\mathbb{D}) \subset \mathbb{R}$.
   - (b) $f : \mathbb{D} \to \mathbb{D}$ such that f is analytic and $f([-1/2, 1/2]) = \{0\}$.
   - (c) $f : \mathbb{C} \to \mathbb{C}$ such that f is analytic and $\mathcal{R}e(f)$ is bounded.
   - (d) $f : \mathbb{C} \to \mathbb{C}$ such that f is analytic and f is bounded on the real and imaginary axes.

5. Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be a linear transformation with the characteristic polynomial $(X-2)^2(X-1)$ and minimal polynomial $(X-2)(X-1)$. Then which of the following are possible matrices for T (*w.r.t.* suitable bases.)?
(a) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

6. Let V be the vector space of polynomials in the variable X with co-efficients in $\mathbb{R}$. Which of the following maps $T : V \to V$ are linear transformations?

   (a) $T(p(X)) = p(X^2)$ for all $p(X) \in V$.
   (b) $T(p(X)) = (p(X))^2$ for all $p(X) \in V$.
   (c) $T(p(X)) = X^2 p(X)$ for all $p(X) \in V$.
   (d) $T(p(X)) = p(X^2+1)$ for all $p(X) \in V$.

7. For a ring R and an element $a \in R$, we denote the ideal generated by a as $\langle a \rangle$. With this notation, determine which of the following rings are integral domains:

   (a) $\mathbb{Z}[i]/\langle 2 \rangle$.
   (b) $\mathbb{Q}[X]/\langle X^4 - 5X + 4 \rangle$.
   (c) $\mathbb{Z}_5[X]/\langle X^2 + X + 1 \rangle$.
   (d) $\mathbb{Z}[X]/\langle 3 \rangle$

8. Which of the following pairs of groups are isomorphic?

   (a) $(\mathbb{R}, +)$, $(\mathbb{C}, +)$.
   (b) $(\mathbb{R}^*, .)$, $(\mathbb{C}^*, .)$.
   (c) $S_3 \times \mathbb{Z}_4$, S_4.
   (d) $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_{12}$.

9. Let us consider two subspaces in $\mathbb{R}$,

$$X = (0,1) \cup \{2\} \cup (4,5) \cup \{6\} \cup \cdots \cup (4n, 4n+1) \cup \{4n+2\} \cup \cdots$$
$$Y = (0,1] \cup (4,5) \cup \{6\} \cup \cdots \cup (4n, 4n+1) \cup \{4n+2\} \cup \cdots ,$$

with two functions $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ defined as follows:

$$f(x) = \begin{cases} x & \text{if } x \neq 2\ , \\ 1 & \text{if } x = 2\ . \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{x}{2} & \text{if } x \in (0,1]\ , \\ \frac{x-3}{2} & \text{if } x \in (4,5), \\ x-4 & \text{otherwise.} \end{cases}$$

Which of the following statement(s) is(are) true?

(a) The map f is not a continuous bijective map.

(b) The map g is not a continuous bijective map.

(c) The maps f and g are continuous bijective map.

(d) The spaces X and Y are homeomorphic.

10. Consider the closed interval $[0,1]$ in the real line $\mathbb{R}$ and the product space $([0,1]^{\mathbb{N}}, \tau)$, where τ is a topology on $[0,1]^{\mathbb{N}}$. Let $D : [0,1] \longrightarrow [0,1]^{\mathbb{N}}$ be the map defined by $D(x) := (x, x, \cdots, x, \cdots)$ for $x \in [0,1]$.

    Find the correct answer(s). The map D is

    (a) not continuous if τ is the box topology and also not continuous if τ is the product topology.

    (b) continuous if τ is the product topology and also continuous if τ is the box topology.

    (c) continuous if τ is the box topology and not continuous if τ is the product topology.

    (d) continuous if τ is the product topology and not continuous if τ is the box topology.

11. Let $\mathcal{C}_{00} := \{(x_n) : \text{ there exists } m \in \mathbb{N} \text{ such that } x_n = 0 \text{ for all } n \geq m\}$ and let $\|(x_n)\| := \sup\{|x_n| : n \in \mathbb{N}\}$. Let $Y := \{(x_n) \in \mathcal{C}_{00} : \sum_{n=0}^{\infty} x_n = 0\}$. For every $n \geq 1$, we denote by $Y_n := (-1, 1, -1/2, 1/2, -1/3, 1/3, \ldots, -1/n, 1/n, 0, \ldots) \in Y$ and $X_n := (-1, \overbrace{\frac{1}{n}, \cdots, \frac{1}{n}}^{n \text{ times}}, 0, 0, \ldots) \in Y$. Determine which of the following are true:

    (a) (X_n) is a Cauchy sequence, but (Y_n) is not a Cauchy sequence in Y.

    (b) (X_n) is not a Cauchy sequence and (Y_n) is a Cauchy sequence in Y.

    (c) (X_n) and (Y_n) are Cauchy sequences in Y.

    (d) neither (X_n) nor (Y_n) is a Cauchy sequence in Y.