The Müntz–Szász theorem

ABSTRACT

In this talk, we will discuss about the more generalized form of Weierstrass Approximation Theorem by Müntz and Szász. We know from Weierstrass's theorem that every continuous function on $[0,1]$ can be uniformly approximated by polynomials. In functional analytic terms, Weierstrass's theorem states that the collection of polynomials is dense in the Banach space $C[0,1]$ of complex-valued continuous functions on $[0,1]$ (with the norm of a function defined by $\|f\|_\infty = sup\{|f(x)| : x \in [0,1]\}$). The following question is very natural: for what subsets $\mathbf{S}$ of the set of natural numbers is the linear span of $\{x^n : n \in \mathbf{S}\} \cup \{1\}$ dense in $C[0,1]$? (Note that the constant functions must be included, for otherwise every element of the linear span would vanish at zero.)

A remarkable theorem established by Müntz and Szász answers a more general question: it allows non-integral powers as well.

Theorem (The Müntz–Szász theorem): If $\{p_n\}$ is a sequence of distinct positive numbers that is bounded away from zero, and if $\sum_{n=1}^{\infty} \frac{1}{p_n}$ diverges, then the linear span of the collection $\{1, x^{p_n} : x^n \in \mathbb{N}\}$ is dense in $C[0,1]$.